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Sharp growth and distortion theorems for a subclass of biholomorphic mappings[☆]

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ABSTRACT

Let X be a complex Banach space with norm $\|\cdot\|$, and B be the unit ball in X. In this paper, we introduce a class of holomorphic mappings $\mathcal{M}_g$ on B. Let $F(x)$ be a normalized locally biholomorphic mapping on B such that $(DF(x))^{-1}F(x) \in \mathcal{M}_g$. We investigate the sharp growth theorem for $F(x)$. As applications, the sharp distortion theorems for a subclass of starlike mappings are obtained.

1. Introduction

In the case of one complex variable, the following growth and distortion theorem is well-known [1].

Theorem A. *Let f be a normalized univalent holomorphic function on the unit disc D in $\mathbb{C}$. Then*

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}, \quad z \in D, \tag{1}$$

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \quad z \in D. \tag{2}$$

However, in the case of several complex variables, Cartan [2] pointed out that the above theorem does not hold.

Barnard et al. [3] and Chuaqui [4] extended the growth theorem (1) to normalized starlike mappings on the Euclidean unit ball in $\mathbb{C}^n$. Subsequently, many mathematicians have investigated the growth theorem for the subclasses of starlike mappings [5–8].

Concerning the distortion theorem, the situation is quite different. Until now, the distortion theorem for the above mappings is still a conjecture. In [9], Pfaltzgraff and Suffridge obtained a distortion result for a subclass of starlike mappings on the Euclidean unit ball in $\mathbb{C}^n$.

In this paper, we shall obtain a sharp growth theorem for a class of biholomorphic mappings. From it, the sharp distortion theorems for a subclass of starlike mappings are obtained. These results are generalizations of the above results.

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Let X be a complex Banach space with norm $\|\cdot\|$, X^* be the dual space of X, B be the unit ball in X, and D be the unit disc in $\mathbb{C}$. For each $x \in X \setminus \{0\}$, we define $T(x) = \{l_x \in X^* : \|l_x\| \leq 1, l_z(x) = \|x\|\}$. According to the Hahn–Banach theorem, $T(x)$ is nonempty. Let $H(B)$ be the set of all holomorphic mappings from B into X. Notice that, for fixed $x \in X$, $\forall \alpha(\neq 0) \in \mathbb{C}$, when l_x is chosen and fixed, then $\|\frac{|\alpha|}{\alpha} l_x\| = \|l_x\| \leq 1$, and $\frac{|\alpha|}{\alpha} l_x(\alpha x) = \frac{|\alpha|}{\alpha}\alpha l_x(x) = |\alpha|\|x\| = \|\alpha x\|$, so we can set $l_{\alpha x} = \frac{|\alpha|}{\alpha} l_x$. A holomorphic mapping $f : B \to X$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. We say that f is normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X. Let $S(B)$ be the set of all normalized biholomorphic mappings. We say that f is starlike if f is biholomorphic on B and $f(B)$ is starlike with respect to the origin. Let $S^*(B)$ be the set of normalized starlike mappings on B. When $B = D$, the set $S^*(D)$ is denoted by S^*.

Definition 1. Let $g \in H(D)$ be a biholomorphic function such that $g(0) = 1$, $g(\bar{\xi}) = \overline{g(\xi)}$, for $\xi \in D$, $\Re e\, g(\xi) > 0$ on $\xi \in D$, and assume that g satisfies the following conditions for $r \in (0, 1)$:

$$\begin{cases} \min\limits_{|\xi|=r} |g(\xi)| = \min\limits_{|\xi|=r} \Re e\, g(\xi) = g(-r) \\ \max\limits_{|\xi|=r} |g(\xi)| = \max\limits_{|\xi|=r} \Re e\, g(\xi) = g(r). \end{cases} \tag{3}$$

We define $\mathcal{M}_g$ to be the class of mappings given by

$$\mathcal{M}_g = \left\{ p \in H(B) : p(0) = 0, Dp(0) = I, \frac{\|x\|}{l_x(p(x))} \in g(D), x \in B \setminus \{0\}, l_x \in T(x) \right\}.$$

Definition 1 was considered by Kohr [10] on B^n and by Graham et al. [11] on the unit ball with respect to an arbitrary norm on $\mathbb{C}^n$. The set $\mathcal{M}_g$ has been important in the study of certain problems related to Löwner chains on the unit ball in $\mathbb{C}^n$ (see [12]).

Let $S_g^*(B)$ denote the subset of $S^*(B)$ consisting of those normalized locally biholomorphic mappings f such that $[Df(x)]^{-1} f(x) \in \mathcal{M}_g$. When $B = D$, the set $S_g^*(D)$ is denoted by S_g^*.

Definition 2. Let $0 \leq \alpha < 1$. A normalized locally biholomorphic mapping $f \in H(B)$ is said to be starlike of order α if

$$[Df(x)]^{-1} f(x) \in \mathcal{M}_g,$$

where $g(\zeta) = \frac{1+(2\alpha-1)\zeta}{1+\zeta}$, $\zeta \in D$.

We denote by $S_\alpha^*(B)$ the set of all starlike mappings of order α on B. When $B = D$, the set $S_\alpha^*(D)$ is denoted by S_α^*.

The following theorem is about the distortion result for the class of S_α^*.

Theorem B. *If $f \in S_\alpha^*$, then*

$$\frac{1-(1-2\alpha)|z|}{(1+|z|)^{3-2\alpha}} \leq |f'(z)| \leq \frac{1+(1-2\alpha)|z|}{(1-|z|)^{3-2\alpha}}, \quad z \in D.$$

These estimates are sharp. Equality holds for $f(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$, $z \in D$.

2. Preliminaries

In order to prove the desired results, we first give some lemmas.

Lemma 1 ([13]). *Suppose that $x(t) : [0, 1] \to X$ is differentiable at the point s which belongs to $[0, 1]$, and $\|x(t)\|$ is differentiable at the point s with respect to t. Then*

$$\Re e \left[T_{x(t)} \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t} \right) \right] \bigg|_{t=s} = \frac{\mathrm{d}(\|x(t)\|)}{\mathrm{d}t} \bigg|_{t=s}.$$

Lemma 2 ([14]). *Suppose that F is a starlike mapping on B, $x \in B \setminus \{0\}$, $x(t) = F^{-1}(tF(x))(0 \leq t \leq 1)$. Then*

(a) *$\|x(t)\|$ is strictly increasing on $[0, 1]$ with respect to t;*
(b) *$\|F(x)\| = \lim_{t\to 0} \frac{\|x(t)\|}{t}$, $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{1}{t}[DF(x(t))]^{-1} F(x(t))$, $t \in (0, 1)$.*

Lemma 3. *If $h \in \mathcal{M}_g$, then*

$$\frac{\|x\|}{g(\|x\|)} \le \Re e\, T_x(h(x)) \le \frac{\|x\|}{g(-\|x\|)} \tag{4}$$

for all $x \in B$.

Proof. Fix $x \in B \setminus \{0\}$, and denote $x_0 = \frac{x}{\|x\|}$. Let $p : D \longrightarrow \mathbb{C}$ be given by

$$p(\xi) = \begin{cases} \dfrac{\xi}{T_x(h(\xi x_0))}, & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

Then $p \in H(D)$, $p(0) = g(0) = 1$, and since $h \in \mathcal{M}_g$, we deduce that

$$p(\xi) = \frac{\xi}{T_x(h(\xi x_0))} = \frac{\xi}{T_{x_0}(h(\xi x_0))} = \frac{\|\xi x_0\|}{T_{\xi x_0}(h(\xi x_0))} \in g(D), \quad \xi \in D.$$

Let $\psi(\xi) = \frac{1}{p(\xi)}$. This implies that $\psi(\xi) \in \frac{1}{g}(D)$ for all $\xi \in D$. Since $\psi(0) = \frac{1}{g}(0) = 1$, we have $\psi \prec \frac{1}{g}$, and from the subordination principle it follows that $\psi(rD) \subseteq \frac{1}{g}(rD)$, $r \in (0, 1)$, where $rD = \{z \in \mathbb{C} : |z| < r\}$. On the other hand, combining the maximum and minimum principles for harmonic functions with (3), we deduce that

$$\frac{1}{g(|\xi|)} \le \Re e\, \psi(\xi) \le \frac{1}{g(-|\xi|)}, \quad \xi \in D.$$

Setting $\xi = \|x\|$ in the above relation, we obtain (4), as desired. This completes the proof. □

3. Main results and their proofs

Theorem 1. *Suppose that $f \in S$ and $F(x) = x\frac{f(l_u(x))}{l_u(x)}$, where u is the unit vector of X. Then $F \in S_g^*(B)$ if and only if $f \in S_g^*$.*

Proof. Denote $h(x) = \frac{f(l_u(x))}{l_u(x)}$. Since $F(x) = h(x)x$, we have $DF(x)\eta = (Dh(x)\eta)x + h(x)\eta$, $\eta \in X$. Also, since $f \in S$, we obtain $\frac{f'(l_u(x))l_u(x)}{f(l_u(x))} \neq 0$, $x \in B$. Straightforward calculation shows that $\frac{Dh(x)x}{h(x)} = \frac{f'(l_u(x))l_u(x)}{f(l_u(x))} - 1$; hence $\frac{Dh(x)x}{h(x)} + 1 \neq 0$. It is not difficult to check that

$$(DF(x))^{-1}\eta = \frac{1}{h(x)}\left[\eta - \frac{(Dh(x)\eta)x}{h(x) + Dh(x)x}\right], \quad \eta \in X.$$

So F is a normalized locally biholomorphic mapping on B, and

$$(DF(x))^{-1}F(x) = \frac{1}{h(x)}\left[x - \frac{(Dh(x)x)x}{h(x) + Dh(x)x}\right]\frac{f(l_u(x))}{l_u(x)} = \frac{f(l_u(x))}{f'(l_u(x))l_u(x)}x. \tag{5}$$

From (5), we have

$$\frac{\|x\|}{l_x[(DF(x))^{-1}F(x)]} = \frac{f'(l_u(x))l_u(x)}{f(l_u(x))},$$

as desired. This completes the proof. □

In [7], Hamada and Honda have recently obtained a sharp growth result for mappings in the family $S^*_{g,\,k+1}(B)$ for which $z = 0$ is a zero of order $k+1$ for $f(z) - z$, and g satisfies a slightly different assumption than that in Definition 1. Stimulated by [7], we obtain the following sharp growth theorem.

Theorem 2. *Let $F : B \longrightarrow X$ be a normalized locally biholomorphic mapping. If $F \in S_g^*(B)$, then*

$$\|x\| \exp \int_0^{\|x\|} [g(-y) - 1]\,\frac{\mathrm{d}y}{y} \le \|F(x)\| \le \|x\| \exp \int_0^{\|x\|} [g(y) - 1]\,\frac{\mathrm{d}y}{y}, \quad x \in B. \tag{6}$$

Proof. Since $F \in S_g^*(B)$, we deduce from Lemma 3 that

$$\frac{\|x\|}{g(\|x\|)} \le \Re e\, T_x(DF(x))^{-1}F(x) \le \frac{\|x\|}{g(-\|x\|)} \tag{7}$$

for all $x \in B$. Fix $x \in B \setminus \{0\}$, let $x(t) = F^{-1}(tF(x))(0 \le t \le 1)$. According to (a) of Lemma 2, we obtain that $\|x(t)\|$ is strictly increasing on $[0, 1]$. Hence, $\|x(t)\|$ is differentiable on $[0, 1]$ a.e. From Lemmas 1, 2(b) and (7), we deduce that, for $t \in (0, 1]$,

$$\frac{\|x(t)\|}{g(\|x(t)\|)} \le t\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t} \le \frac{\|x(t)\|}{g(-\|x(t)\|)}, \tag{8}$$

and we may rewrite (8) as

$$\frac{g(-\|x(t)\|)}{\|x(t)\|}\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t} \le \frac{1}{t} \le \frac{g(\|x(t)\|)}{\|x(t)\|}\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t}.$$

Integrating both sides of the above inequalities with respect to t and making a change of variable, we obtain

$$\int_{\|x(\varepsilon)\|}^{\|x\|} \frac{g(-y)\mathrm{d}y}{y} = \int_{\varepsilon}^{1} \frac{g(-\|x(t)\|)}{\|x(t)\|}\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t}\mathrm{d}t \le \int_{\varepsilon}^{1}\frac{1}{t}\mathrm{d}t,$$

and

$$\int_{\|x(\varepsilon)\|}^{\|x\|} \frac{g(y)\mathrm{d}y}{y} = \int_{\varepsilon}^{1} \frac{g(\|x(t)\|)}{\|x(t)\|}\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t}\mathrm{d}t \ge \int_{\varepsilon}^{1}\frac{1}{t}\mathrm{d}t,$$

where $0 < \varepsilon < 1$. It is elementary to verify that

$$\log\frac{\|x(\varepsilon)\|}{\varepsilon} \ge \int_{\|x(\varepsilon)\|}^{\|x\|}[g(-y)-1]\frac{\mathrm{d}y}{y} + \log\|x\|, \tag{9}$$

and

$$\log\frac{\|x(\varepsilon)\|}{\varepsilon} \le \int_{\|x(\varepsilon)\|}^{\|x\|}[g(y)-1]\frac{\mathrm{d}y}{y} + \log\|x\|. \tag{10}$$

If we now let $\varepsilon \longrightarrow 0+$ in the above inequalities (9), (10) and use Lemma 2(b), we have

$$\|x\|\exp\int_0^{\|x\|}[g(-y)-1]\frac{\mathrm{d}y}{y} \le \|F(x)\| \le \|x\|\exp\int_0^{\|x\|}[g(y)-1]\frac{\mathrm{d}y}{y}, \quad x \in B,$$

as claimed. This completes the proof of Theorem 1. □

Remark 1. The estimations of Theorem 2 are sharp. To see this, let $b \in S_g^*$ be defined by $b(0) = b'(0) - 1 = 0$ and

$$\frac{\xi b'(\xi)}{b(\xi)} = g(\xi), \quad \xi \in D. \tag{11}$$

Also, for $u \in \partial B$, let

$$F_u(x) = \frac{f(l_u(x))}{l_u(x)}x, \quad x \in B. \tag{12}$$

In view of Theorem 1, $F_u \in S_g^*(B)$. From (11), we obtain the following equivalent formulation of Theorem 2.

$$\exp\int_0^{\|x\|}\left[\frac{y\tilde{f}'(y)}{\tilde{f}(y)} - 1\right]\frac{\mathrm{d}y}{y} \le \frac{\|F(x)\|}{\|x\|} \le \exp\int_0^{\|x\|}\left[\frac{yf'(y)}{f(y)} - 1\right]\frac{\mathrm{d}y}{y}$$

for $x \in B$, where $\tilde{f}(\xi) = -f(-\xi)$. Then, we have

$$\exp\left[\log\frac{\tilde{f}(\|x\|)}{\|x\|} - \log\tilde{f}'(0)\right] \le \frac{\|F(x)\|}{\|x\|} \le \exp\left[\log\frac{f(\|x\|)}{\|x\|} - \log f(0)\right]$$

for $x \in B$, since $\tilde{f}(y), f(y) > 0$ for $y > 0$. We deduce that

$$-f(-\|x\|) \le \|F(x)\| \le f(\|x\|), \quad x \in B. \tag{13}$$

Next, we will show that the estimations (13) are sharp. Let $F_u \in S_g^*(B)$ be as in (12). Since $\|F_u(ru)\| = f(r)$ and $\|F_u(-ru)\| = |f(-r)|$, the equalities of the estimations (13) hold. The equivalence of (6) and (13) implies that the estimations (6) are sharp. This completes the proof. □

Theorem 3. *Let $g : D \to \mathbb{C}$ be a convex function which satisfies the conditions of Definition 1. Suppose that $f_j \in S_g^*$, $\lambda_j \geq 0,\ j = 1, 2, \dots, n$, and $\sum_{j=1}^n \lambda_j = 1$. Then*

$$F \in S_g^*(B),$$

where $F(x) = x\prod_{j=1}^n \left(\frac{f_j(l_{u_j}(x))}{l_{u_j}(x)}\right)^{\lambda_j}$, $x \in B,\ u_j \in X,\ \|u_j\| = 1,\ (j = 1, \dots, n)$.

Proof. Let $h(x) = \prod_{j=1}^n \left(\frac{f_j(l_{u_j}(x))}{l_{u_j}(x)}\right)^{\lambda_j}$. Then $DF(x)\eta = h(x)\left(\eta + \frac{(Dh(x)\eta)x}{h(x)}\right)$, $\eta \in X$. Also

$$\frac{Dh(x)x}{h(x)} = \sum_{j=1}^n \lambda_j \left(\frac{f_j'(x_j)x_j}{f_j(x_j)} - 1\right) = \sum_{j=1}^n \lambda_j \frac{f_j'(x_j)x_j}{f_j(x_j)} - 1,$$

where $x_j = l_{u_j}(x), j = 1, \dots, n$. Since $f_j \in S_g^*(j = 1, 2, \dots, n)$, we have

$$\Re e\left[1 + \frac{Dh(x)x}{h(x)}\right] = \sum_{j=1}^n \lambda_j \Re e\left[\frac{f_j'(x_j)x_j}{f_j(x_j)}\right] > 0, \quad x \in B.$$

Therefore, $Dh(x)x + h(x) \neq 0$. It is not difficult to check that

$$(DF(x))^{-1}\eta = \frac{1}{h(x)}\left(\eta - \frac{(Dh(x)\eta)x}{Dh(x)x + h(x)}\right), \quad \eta \in X.$$

So $F(x)$ is a normalized locally biholomorphic mapping on B.

Straightforward calculation shows that

$$\frac{\|x\|}{l_x[(DF(x))^{-1}F(x)]} = \frac{Dh(x)x + h(x)}{h(x)x} = \sum_{j=1}^n \lambda_j \frac{f_j'(x_j)x_j}{f_j(x_j)}.$$

On the other hand, since $\frac{f_j'(x_j)x_j}{f_j(x_j)} \in g(D)$, and $g(D)$ is convex, we have

$$\sum_{j=1}^n \lambda_j \frac{f_j'(x_j)x_j}{f_j(x_j)} \in g(D).$$

This completes the proof. □

Theorem 4. *Let $g : D \to \mathbb{C}$ be a convex function which satisfies the conditions of Definition 1. Suppose that $f_j \in S_g^*$, $\lambda_j \geq 0,\ j = 1, 2, \dots, n$, and $\sum_{j=1}^n \lambda_j = 1$. Then*

$$g(-\|z\|)\exp n\int_0^{\|z\|} [g(-y) - 1] \leq |\det J_F(z)| \leq g(\|z\|)\exp n\int_0^{\|z\|} [g(y) - 1]\frac{\mathrm{d}y}{y}, \tag{14}$$

where $z = (z_1, \dots, z_n)' \in B$, $F(z) = z\prod_{j=1}^n \left(\frac{f_j(l_{u_j}(z))}{l_{u_j}(z)}\right)^{\lambda_j}$, $\|u_j\| = 1,\ (j = 1, \dots, n)$, B is the unit ball of $\mathbb{C}^n$ with arbitrary norm $\|\cdot\|$, and $J_F(z)$ is the Jacobi matrix of $F(z)$.

Proof. Denote

$$h(z) = \prod_{j=1}^n \left(\frac{f_j(l_{u_j}(z))}{l_{u_j}(z)}\right)^{\lambda_j}, \quad z \in B.$$

Then

$$J_F(z) = h(z)\left(I + z\sum_{j=1}^n \lambda_j \left(\frac{f_j'(l_{u_j}(z))}{f_j(l_{u_j}(z))} - \frac{1}{l_{u_j}(z)}\right) l_{u_j}(\cdot)\right),$$

where $l_{u_j}(z)$ is a row vector of $\mathbb{C}^n$. Hence

$$\begin{aligned} |\det J_F(z)| &= |h(z)|^n \left|1 + \sum_{j=1}^n \lambda_j \left(\frac{f_j'(l_{u_j}(z))}{f_j(l_{u_j}(z))} - \frac{1}{l_{u_j}(z)}\right) l_{u_j}(z)\right| \\ &= \prod_{j=1}^n \left|\frac{f_j(l_{u_j}(z))}{l_{u_j}(z)}\right|^{n\lambda_j} \left|\sum_{j=1}^n \lambda_j \frac{f_j'(l_{u_j}(z))l_{u_j}(z)}{f_j(l_{u_j}(z))}\right|. \end{aligned}$$

From the maximum (minimum) principle for harmonic functions and the fact that $|l_{u_j}(z)| \le \|z\| (j = 1, \ldots, n)$, we obtain

$$\left|\sum_{j=1}^{n} \lambda_j \frac{f_j'(l_{u_j}(z)) l_{u_j}(z)}{f_j(l_{u_j}(z))}\right| \ge \sum_{j=1}^{n} \lambda_j \Re e \frac{f_j'(l_{u_j}(z)) l_{u_j}(z)}{f_j(l_{u_j}(z))} \ge g(-\|z\|), \tag{15}$$

and

$$\left|\sum_{j=1}^{n} \lambda_j \frac{f_j'(l_{u_j}(z)) l_{u_j}(z)}{f_j(l_{u_j}(z))}\right| \le \sum_{j=1}^{n} \lambda_j \left|\frac{f_j'(l_{u_j}(z)) l_{u_j}(z)}{f_j(l_{u_j}(z))}\right| \le g(\|z\|). \tag{16}$$

On the other hand, by Theorems 2 and 3, we have

$$\exp \int_0^{\|z\|} [g(-y) - 1] \frac{\mathrm{d}y}{y} \le \prod_{j=1}^{n} \left|\frac{f_j(l_{u_j}(z))}{l_{u_j}(z)}\right|^{\lambda_j} \le \exp \int_0^{\|z\|} [g(y) - 1] \frac{\mathrm{d}y}{y}, \quad z \in B. \tag{17}$$

From (15)–(17), the result follows. This completes the proof. □

Remark 2. The estimations of Theorem 4 are sharp. Let $f \in S_g^*$ and F_u be defined by (11) and (12), respectively; we obtain the following equivalent formulation of Theorem 4.

$$\frac{\|z\|\tilde{f}'(\|z\|)}{\tilde{f}(\|z\|)} \exp n \int_0^{\|z\|} \left[\frac{y\tilde{f}'(y)}{\tilde{f}(y)} - 1\right] \frac{\mathrm{d}y}{y} \le |\det J_F(z)| \le \frac{\|z\| f'(\|z\|)}{f(\|z\|)} \exp n \int_0^{\|z\|} \left[\frac{y f'(y)}{f(y)} - 1\right] \frac{\mathrm{d}y}{y}$$

for $z \in B$, where $\tilde{f}(\xi) = -f(-\xi)$. Then, we have

$$\frac{\|z\|\tilde{f}'(\|z\|)}{\tilde{f}(\|z\|)} \exp n \left[\log \frac{\tilde{f}(\|z\|)}{\|z\|} - \log \tilde{f}'(0)\right] \le |\det J_F(z)| \le \frac{\|z\| f'(\|z\|)}{f(\|z\|)} \exp n \left[\log \frac{f(\|z\|)}{\|z\|} - \log f(0)\right]$$

for $z \in B$, since $\tilde{f}(y), f(y) > 0$ for $y > 0$. We deduce that

$$\frac{-\|z\| f'(-\|z\|)}{f(-\|z\|)} \left(\frac{-f(-\|z\|)}{\|z\|}\right)^n \le |J_F(z)| \le \frac{\|z\| f'(\|z\|)}{f(\|z\|)} \left(\frac{f(\|z\|)}{\|z\|}\right)^n, \quad z \in B. \tag{18}$$

Taking $z = ru$ or $z = -ru$, then the equalities of the estimations (18) hold for $\lambda_1 = 1$, $\lambda_j = 0 (j = 2, \ldots, n)$, and $u_j = u$, $f_j = f (j = 1, \ldots, n)$. The equivalence of (14) and (18) implies that the estimations (14) are sharp. This completes the proof. □

Theorem 5. *Suppose that $f_j \in S_g^*$, $\lambda_j \ge 0$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} \lambda_j = 1$. Then*

$$\|x\| g(-\|x\|) \exp \int_0^{\|x\|} [g(-y) - 1] \le \|DF(x)x\| \le \|x\| g(\|x\|) \exp \int_0^{\|x\|} [g(y) - 1] \frac{\mathrm{d}y}{y}, \quad x \in B,$$

where $F(x) = x \prod_{j=1}^{n} \left(\frac{f_j(l_{u_j}(x))}{l_{u_j}(x)}\right)^{\lambda_j}$, $u_j \in B$, and $\|u_j\| = 1$, $(j = 1, \ldots, n)$.

Proof. Denote

$$h(x) = \prod_{j=1}^{n} \left(\frac{f_j(l_{u_j}(x))}{l_{u_j}(x)}\right)^{\lambda_j}, \quad x \in B.$$

Straightforward computation shows that

$$DF(x)x = h(x)x + (Dh(x)x)x = x \prod_{j=1}^{n} \left(\frac{f_j(x_j)}{x_j}\right)^{\lambda_j} \sum_{j=1}^{n} \lambda_j \frac{f_j'(x_j) x_j}{f_j(x_j)}, \tag{19}$$

where $x_j = l_{u_j}(x)$ $(j = 1, \ldots, n)$.

From (15)–(17) and (19), we have the desired result. This completes the proof. □

Remark 3. The estimations of Theorem 5 are sharp. The proof of sharpness is similar to that of Theorem 4, so we omit it.

For $g(\zeta) = \frac{1+(2\alpha-1)\zeta}{1+\zeta}$, $\zeta \in D$, from Theorems 4 and 5, we obtain the following corollaries.

Corollary 1. *Suppose that $f_j \in S_\alpha^*$ $(0 \le \alpha < 1)$, $\lambda_j \ge 0$ $(j = 1, \ldots, n)$, and $\sum_{j=1}^n \lambda_j = 1$. Then*

$$\frac{1-(1-2\alpha)\|z\|}{(1+\|z\|)^{1+2(1-\alpha)n}} \le |\det J_F(z)| \le \frac{1+(1-2\alpha)\|z\|}{(1-\|z\|)^{1+2(1-\alpha)n}}, \tag{20}$$

where $z = (z_1, \ldots, z_n)' \in B$, $F(z) = z\prod_{j=1}^n \left(\frac{f_j(l_{u_j}(z))}{l_{u_j}(z)}\right)^{\lambda_j}$, $\|u_j\| = 1$, $(j = 1, \ldots, n)$, B is the unit ball of $\mathbb{C}^n$ with arbitrary norm $\|\cdot\|$, and $J_F(z)$ is the Jacobi matrix of $F(z)$. These estimations are sharp.

Corollary 2. *Suppose that $f_j \in S_\alpha^*$ $(0 \le \alpha < 1)$, $\lambda_j \ge 0$ $(j = 1, \ldots, n)$, and $\sum_{j=1}^n \lambda_j = 1$. Then*

$$\frac{(1-(1-2\alpha)\|x\|)\|x\|}{(1+\|x\|)^{3-2\alpha}} \le \|DF(x)x\| \le \frac{(1+(1-2\alpha)\|x\|)\|x\|}{(1-\|x\|)^{3-2\alpha}}, \quad x \in B \tag{21}$$

where $F(x) = x\prod_{j=1}^n \left(\frac{f_j(l_{u_j}(x))}{l_{u_j}(x)}\right)^{\lambda_j}$, $\|u_j\| = 1$, $(j = 1, \ldots, n)$. These estimations are sharp.

Remark 4. When $n = 1$, $B = D$, Corollaries 1 and 2 are the same as Theorem B.

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